2 Modules

Exercises 2.1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$, if m, n coprime.

Solution. If m, n coprime then n is a unit in $\mathbb{Z}_m$, so

$$x \otimes y = n^{-1}x \otimes ny = 0$$

Hence $(\mathbb{Z}/mZ) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$, 'cause it's generated by all $x \otimes y$. □

Exercises 2.2. Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module, show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

Solution. Obviously $\mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ is an exact sequence, so is $\mathfrak{a} \otimes M \to A \otimes M \to (A/\mathfrak{a}) \otimes M \to 0$. But $\mathfrak{a} \otimes M \cong \mathfrak{a}M$ and $A \otimes M \cong M$, and the first arrow is the inclusion map, so $(A/\mathfrak{a}) \otimes M \cong (M/\mathfrak{a}M)$. □

Exercises 2.3. Ler A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

Solution. Let $\mathfrak{m}$ be the maximal ideal of A and $k = A/\mathfrak{m}$ be the residue field of A. Let M_k denote $k \otimes_A M = (M/\mathfrak{m}M)$, then by Nakayama Lemma, $M_k = 0 \to M = 0$. So we have $M \otimes_A N = 0 \implies (M \otimes_A N)_k = 0 \implies M_k \otimes_k N_k = 0$. But M_k and N_k are vector spaces over field k, so $M_k \otimes_k N_k = 0$ implies $M_k = 0$ or $N_k = 0$, hence $M = 0$ or $N = 0$. □

Exercises 2.4. Let $M_i (i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\iff$ each M_i is flat.

Solution. $M = \bigoplus_{i \in I} M_i$ is flat $\iff$ for all injective $f : N \to N'$, $f \otimes (\bigoplus_{i \in I} 1_{M_i}) = \bigoplus_{i \in I} (f \otimes 1_{M_i}) : N \otimes M \to N' \otimes M$ is injective. And $\bigoplus_{i \in I} f_i$ is injective if and only if each f_i is injective, so qed. □

Exercises 2.5. Ler $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Solution. As a A-module, $A[x] \cong \bigoplus_{n=0}^{\infty} A$, so by Exercise2.4 $A[x]$ is flat (since A is flat). □

Exercises 2.6. For any A-module M, let $M[x]$ denote the set of all polynomials in x with coefficients in M. Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module.

Show that $M[x] \cong A[x] \otimes_A M$.

Solution. By define $(\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{k} m_j x^j) = \sum_{i=0}^{n} \sum_{j=0}^{k} a_i m_j x^{i+j}$, trivially the module axioms hold here.

Consider a map $\phi : M[x] \to A[x] \otimes_A M$ defined by $\phi(mx^i) = x^i \otimes m$, then it's a well-defined $A[x]$-module homomorphism. If we define $\bar{\psi} : A[x] \times M \to M[x]$ by $\bar{\psi}(\sum_i a_i x^i, m) = \sum_i (a_i m) x^i$, then it's clearly A-bilinear, so induces an A-module homomorphism $\psi : A[x] \otimes_A M \to M[x]$. It's easy to prove ϕ and ϕ are inverse, so $A[x] \otimes_A M \cong M[x]$. □

Exercises 2.7. Let $\mathfrak{p}$ be a prime ideal in A, show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Solution. Consider map $\phi : A[x] \to (A/\mathfrak{p})[x]$, then $\operatorname{Ker}\phi = \mathfrak{p}[x]$. Then $\mathfrak{p}[x]$ is prime since $(A/\mathfrak{p})[x]$ is an integral domain.

If $\mathfrak{m}$ is maximal, then $(A/\mathfrak{m})[x]$ doesn't have to be a field, so $\mathfrak{m}[x]$ is not maximal in general. For a counterexample, $2\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$, but $(2\mathbb{Z})[x] \subseteq (2, x)$ is not maximal. □

Exercises 2.8.

i) If M and N are flat A-modules, then so is $M \otimes_A N$.

ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution.

i) If $U \to V \to W$ is an exact sequence, then so is $(U \otimes M) \to (V \times M) \to (W \times M)$, hence $(U \otimes M) \otimes N \to (V \otimes M) \otimes N \to (W \otimes M) \otimes N$.

   but $(U \otimes M) \otimes N \cong U \otimes (M \otimes N)$, qed.

ii) Let $j : M \to M'$ be an injective A-module homomorphism. Since B is flat, $(\mathrm{id}_B \otimes_A j) : (B \otimes_A M) \to (B \otimes_A M')$ is injective. Consider $(\mathrm{id}_B \otimes_A j)$ as a B-module homomorphism, then since N is flat, $\mathrm{id}_N \otimes_B (\mathrm{id}_B \otimes_A j)$ is injective.

   By associativity of tensor product, and $N \otimes_B B \cong N$, we have $\mathrm{id}_N \otimes_A j$ injective, hence N is flat as A-module.

□

Exercises 2.9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequnence of A-modules. If M' and M'' are finitely generated, so is M.

Solution. Let $f : M' \to M$ and $g : M \to M''$ be the maps in the sequence. If $x_1, \dots, x_n$ generate M' and $y_1, \dots, y_m$ generate M''. For each y_i select an element $q_i \in M$ such that $g(q_i) = y_i$, then since $M = \bigcup_{i=1}^m (q_i + \operatorname{Ker} g) = \operatorname{Im} f + \sum_{i=1}^m (q_i)$, and $\operatorname{Im} f$ is generated by $p_i = f(x_i)$, so M is generated by $p_1, \dots, p_n, q_1, \dots, q_m$. □

Exercises 2.10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution. If $\bar{u} : M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then forall $y \in N$ there exists $x \in M$ such that $u(x) - y \in \mathfrak{a}N$. That means, $N = \operatorname{Im} u + \mathfrak{a}N$. So by Nakayama Lemma, $N = \operatorname{Im} u$, i.e. u is surjective. □

Exercises 2.11. Let A be a ring $\neq 0$. Show that $A^m \cong A^n \implies m = n$.

Solution. let $\mathfrak{m}$ be a maximal ideal of A, and $\phi : A^m \to A^n$ an isomorphism, then $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism between two $A/\mathfrak{m}$-vector spaces, hence the dim of two space are same, i.e. $m = n$. □

Exercises 2.12. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\operatorname{Ker}\phi$ is finitely generated.

Solution. Let $x_1, \ldots, x_n$ be a set of generators of A^n, and $y_1, \ldots, y_n \in M$ such that $\phi(y_i) = x_i$. Let M' be the submodule generating by $y_1, \ldots, y_n$, then clearly $M' \cap \operatorname{Ker}\phi = 0$, and for all $t \in M$ there exists $y \in M'$ such that $f(t) = f(y)$, hence $M' + \operatorname{Ker}\phi = M$. Summarize results above we get $M \cong M' \oplus \operatorname{Ker}\phi$. Since M is finitely generated, $\operatorname{Ker}\phi$ must be finitely generated too. □

Exercises 2.13. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution. Consider the quotient map $B \otimes_A N \to B \otimes_B N$. Since $B \otimes_B N \cong N$, we have $h : N_B \to N$ which maps $b \otimes y$ to by.

Now $h \circ g = \mathrm{id}_N$, so g is injective. Consider map $\phi : N_B \to N \oplus \operatorname{Ker} h$ defined by $\phi = h \oplus (\mathrm{id}_{N_B} - g \circ h)$. ($h(x - g(h(x))) = h(x) - h(x) = 0$ so the second part of image of ϕ is actually $\operatorname{Ker} h$). We will prove ϕ is an isomorphism so $N_B \cong N \oplus \operatorname{Ker} h$.

If $\phi(x) = 0$, i.e. $h(x) = 0$ and $x - g(h(x)) = 0$, obviously $x = 0$, so ϕ is injective. For any $y \in N$ and $x_0 \in \operatorname{Ker} h$, let $x = x_0 + g(y)$, then $h(x) = y$ and $x - g(h(x)) = x_0$, hence ϕ is injective. All in all we have ϕ is an isomorphism so $N_B \cong N \oplus \operatorname{Ker} h$. □

Direct limits

Exercises 2.14. A partially ordered set I is said to be a *direct* set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let A be a ring, let I be a direct set and let $(M_i)_{i \in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

i) μ_{ii} is the identity mapping of M_i for all $i \in I$;

ii) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.

Then the modules M_i and homomorphisms μ_{ij} are said to form a *direct system* $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the *direct limit* of the direct system $\mathbf{M}$. Let C be the direct sum of M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.

The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$ is called the *direct limit* of the direct system $\mathbf{M}$, and is written $\varinjlim M_i$. From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.

Solution. No exercise here. For the last sentence, $\mu_i(x) - \mu_j(\mu_{ij}(x)) = \mu(x - \mu_{ij}(x))$, but $x - \mu_{ij}(x) \in D = \operatorname{Ker}\mu$. □

Exercises 2.15. In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Solution. Any element of M can be written in form $\sum_{i \in I} \mu_i(x_i)$ with only finite x_i nonzero. But for any i, j there exists k such $i, j \leq k$, so $\mu_i(x_i) + \mu_j(x_j) = \mu_k(\mu_{ik}(x_i) + \mu_{jk}(x_j))$, hence any element of M can be written in form $\mu_k(x_k)$ for some k.

If $\mu_i(x_i) = 0$, i.e. $x_i \in \operatorname{Ker} \mu = D$, then we have

$$x_i = \sum_{j,k \in I} (y_j - \mu_{jk}(y_j)) = \sum_{j \in I} z_j$$

where the sum contains only finite nonzero terms, and z_j is projection to M_j. But $x_i \in M_i$ and the equation above is in a direct sum, so all elements $z_j = 0$ except $z_i = x_i$. Select an index $p \in I$ which $\geq$ any j, k appearing here, then

$$\mu_{ip}(x_i) = \sum_{j \in I} \mu_{jp}(z_j) = \sum_{j,k \in I} (\mu_{jp}(y_j) - (\mu_{kp} \circ \mu_{jk})(y_j)) = 0$$

'Cause $\mu_{jp} = \mu_{kp} \circ \mu_{jk}$ for any $j \leq k \leq p$. □

Exercises 2.16. Show that the direct limit is charactered (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Solution. First prove (M, μ_i) constructed here satisfies this condition.

For any N and $\alpha_i : M_i \to N$ satisfies $\alpha_i = \alpha_j \circ \mu_{ij}$, define $\beta : C \to N$ by $\beta(\sum_{i \in I} x_i) = \sum_{i \in I} \alpha_i(x_i)$, then for all $i \leq j$ and $x_i \in M_i$ we have $\beta(x_i - \mu_{ij}(x_j)) = 0$, hence $D \subseteq \operatorname{Ker} \beta$, so β induce an A-homomorphism $\alpha : M \to N$ and $\alpha_i = \alpha \circ \mu_i$ for any $i \in I$. Since all elements in M can be written in the form $\mu_i(x_i)$, the map α is then unique.

If (M', μ_i') is another system satisfying the condition, let $N = M$ and $\alpha_i = \mu_i$, there a unique homomorphism $\alpha : M' \to M$ such that $\mu_i = \alpha \circ \mu_i'$. In the other direction there also exists a homomorphism $\beta : M \to M'$ such that $\mu_i' = \beta \circ \mu_i$, so $\mu_i = \alpha \circ \beta \circ \mu_i$ for any $i \in I$. Again let $N = M$ and $\alpha_i = \mu_i$, there exists a unique homomorphism $\gamma : M \to M$ such that $\mu_i = \gamma \circ \mu_i$ for any i. But both $\alpha \circ \beta$ and id_M meet the requirement of γ, so $\mathrm{id}_M = \alpha \circ \beta$; and $\mathrm{id}_{M'} = \beta \circ \alpha$ vice versa. Hence α and β are inverse, and $M \cong M'$. □

Exercises 2.17. Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j. Show that

$$\varinjlim M_i = \sum M_i = \bigcup M_i$$

Solution. $\sum M_i = \bigcup M_i$ is obviously hold since for all i, j there exists some k, $M_i + M_j \subseteq M_k$.

We show $\varinjlim M_i \cong \bigcup M_i$ by show $\bigcup M_i$ have the universal property in the previous exercise. If given (N, α_i) such that $\alpha_i = \alpha_j \circ \mu_{ij}$ for any pair $M_i \subseteq M_j$, then for all pair of indices i, j and element $x \in M_i \cap M_j$ there exists $M_k \supseteq M_i \cup M_j$, so $\alpha_i(x) = \alpha_k(x) = \alpha_j(x)$.

Therefore we can define $\alpha : \bigcup M_i \to N$ that agrees each α_i over M_i. Since any element belonging to $\bigcup M_i$ also belongs to some M_i, so the homomorphism here is unique.

By the previous exercise, we have then $\varinjlim M_i \cong \bigcup M_i$. □

Exercises 2.18. Let $\mathbf{M} = (M_i, \mu_{ij}), \mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M, \nu_i : N_i \to N$ the associated homomorphisms.

A *homomorphism* $\Phi : \mathbf{M} \to \mathbf{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leq j$. Show that Φ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Solution. Let $\phi_i' = \nu_i \circ \phi_i : M_i \to N$, then $\phi_j' \circ \mu_{ij} = \nu_j \circ \nu_{ij} \circ \phi_i = \phi_i'$ whenever $i \leq j$. Hence there exists a unique homomorphism $\phi : M \to N$ such that $\phi \circ \mu_i = \phi_i' = \nu_i \circ \phi_i$ for all $i \in I$. □

Exercises 2.19. A sequence of direct systems and homomorphism

$$\mathbf{M} \to \mathbf{N} \to \mathbf{P}$$

is *exact* if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact.

Solution. Let $\phi_i : M_i \to N_i, \psi_i : N_i \to P_i$ be the corresponding homomorphisms, $\mu_{ij}, \nu_{ij}, \pi_{ij}$ be the homomorphisms in system $\mathbf{M}, \mathbf{N}, \mathbf{P}$, and $\phi : M \to N, \psi : N \to P$ the induced homomorphisms.

First we show $\psi \circ \phi = 0$. For any $x \in M$, there exists $i \in I$ and $x_i \in M_i$ such that $x = \mu_i(x_i)$, and then

$$\psi(\phi(x)) = (\psi \circ \phi \circ \mu_i)(x) = (\pi_i \circ \psi_i \circ \phi_i)(x) = \pi_i(0) = 0$$

Then if $y \in \operatorname{Ker} \psi$, again $y = \nu_i(y_i)$ for some $i \in I$ and $y_i \in N_i$, hence $0 = \psi(\nu_i(y_i)) = \pi_i(\psi_i(y_i))$. So there exists $j \geq i$ such that $0 = \pi_{ij}(\psi_i(y_i)) = \psi_j(\nu_{ij}(y_j))$, hence $\nu_{ij}(y_j) \in \operatorname{Ker} \psi_j = \operatorname{Im} \phi_j$, and there exists x_j such that $\phi_j(x_j) = \nu_{ij}(y_j)$. Let $x = \mu_j(x_j)$, we have

$$\psi(x) = \psi(\mu_j(x_j)) = \nu_j(\psi_j(x_j)) = \nu_j(nu_{ij}(y_j)) = \nu_i(y_i) = y$$

Summarize the results above, we have $\operatorname{Ker} \psi = \operatorname{Im} \phi$, hence $M \to N \to P$ is exact. □

Tensor products commute with direct limits

Exercises 2.20. Keeping the tame notation as in Exercise 14, Let N be any A-module, Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \varinjlim (M_i \otimes N)$ be its direct limit.

For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$, hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim (M_i \otimes N) \cong (\varinjlim M_i) \otimes N$$

Solution. Let π_i be the projection map form $M_i \otimes N$ to P.

Define $\phi_{y,i} : M_i \to P$ by $x_i \mapsto \pi_i(x_i \otimes y)$, then clearly $\phi_{y,i} = \phi_{y,j} \circ \mu_{ij}$, so $\phi_{y,i}$ induce a homomorphism $\phi_y : M \to P$. Since every $\phi_{y,i}$ is A-linear over y, by the construction of ϕ_y it's easy to show so is $\phi_y(x)$. So we have a homomorphism $\phi : M \otimes N \to P$ defined by $\phi(x \otimes y) = \phi_y(x)$. We will show that ϕ is the inverse of ψ. We have:

$$\phi(\mu_i(x_i) \otimes y) = \phi_y(\mu_i(x_i)) = \phi_{y,i}(x_i) = \pi_i(x_i \otimes y)$$
$$\psi(\pi_i(x_i \otimes y)) = (\mu_i \otimes 1)(x_i \otimes y) = \mu_i(x) \otimes y$$

Since all $x \in M$ can be written in form $\mu_i(x_i)$ and all $p \in P$ can be written in form $\pi_i(x_i \otimes y)$, it's clearly $\phi \circ \psi = \mathrm{id}_P, \psi \circ \phi = \mathrm{id}_{M \otimes N}$. So

$$\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N \quad \square$$

Exercises 2.21. Let $(A_i)_{i \in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leq j$ in I let $\alpha_{ij} : A_i \to A_j$ be a ring homomorphism, satisfying conditions i) and ii) of Exercise 14. Regarding each A_i as a $\mathbb{Z}$-module we can then form the direct limit $A = \varinjlim A_i$. Show that A inherits a ring structure from the A_i so that the mappings $A_i \to A$ are ring homomorphism. The ring A is the *direct limit* of the system (A_i, α_{ij}).

If $A = 0$ prove that $A_i = 0$ for some $i \in I$.

Solution. Let $\alpha_i : A_i \to A$ be the mappings. In A every element is some $\alpha_i(a_i)$ where $i \in I$ and $a_i \in A_i$. For any $\alpha_i(a_i)$ and $\alpha_j(a_j)$, let k be an index $\geq i, j$, we define $\alpha_i(a_i) \cdot \alpha_j(a_j) = \alpha_k(\alpha_{ik}(a_i)\alpha_{jk}(a_j))$. If there are two indices $k_1, k_2 \geq i, j$, find an index $p \geq k_1, k_2$ we can show that the definition does not depend on the choice of k. The ring axioms are easy to verify, with the identity element be any $\alpha_i(1)$ (they are all equal).

For the second part, if $A = 0$, select an index $i \in I$, then $\alpha_i(1) = 0$. By Exercise 15 there exists $j \geq i$ such that $\alpha_{ij}(1) = 0$. Since α_{ij} is a ring homomorphism, A_j must be 0. $\square$

Exercises 2.22. Let (A_i, α_{ij}) be a direct system of rings and let $\mathfrak{R}_i$ be the nilradical of A_i. Show that $\varinjlim \mathfrak{R}_i$ is the nilradical of $\varinjlim A_i$.

If each A_i is an integral domain, then $\varinjlim A_i$ is an integral domain.

Solution. If $x \in \mathfrak{R}_i$, then clearly $\alpha_{ij}(x_i) \in \mathfrak{R}_j$. So $(\mathfrak{R}_i, \bar{\alpha}_{ij})$ is a direct system where $\bar{\alpha}_{ij}$ is the restriction of α_{ij}.

Let A denote the direct limit of A_i. An element $\mu_i(x_i) \in A$ is nilpotent iff. $\exists n > 0$, $\mu_i(x_i^n) = 0$ iff. $\exists n > 0$ and $j \geq i$ such that $\mu_{ij}(x_i)^n = 0$, i.e. exists $j \geq i$ such that $\mu_{ij}(x_i)$ is nilpotent in A_j. That is, an element $x \in A$ is nilpotent if and only if it can be written in form $\mu_j(x_j)$ where $x_j \in \mathfrak{R}_j$. So the nilradical of A is $\varinjlim \mathfrak{R}_i$, the proposition holds.

For the second part, if $xy = 0 \in A$, then there exists $i \in I$ such that $x = \mu_i(x_i)$ and $y = \mu_i(y_i)$, hence $\mu_i(x_i y_i) = 0$ and there exists $j \geq i$ that $\mu_{ij}(x_i y_i) = 0$. Since A_j is an integral domain, either $\mu_{ij}(x_i)$ or $\mu_{ij}(y_i)$ is zero, so in A either x or y is zero, and A is then an integral domain. $\square$

Exercises 2.23. Let $(B_\lambda)_{\lambda \in \Lambda}$ be a family of A-algebras. For each finite subset of Λ let B_J denote the tensor product (over A) of the B_λ for $\lambda \in J$. If J' is another finite subset of Λ and $J \subseteq J'$, there is a canonical A-algebra homomorphism $B_J \to B_{J'}$. Let B denote the direct limit of the rings B_J as J runs through all finite subsets of Λ. The ring B has a natural A-algebra structure for which the homomorphisms $B_J \to B$ are A-algebra homomorphisms. The A-algebra is the *tensor product* of the family $(B_\lambda)_{\lambda\in\Lambda}$.

Solution. For $J = \{\lambda_1, \dots, \lambda_n\}$ and $J' = J \cup \{\lambda_{n+1}, \dots \lambda_{n+m}\}$ we have a canonical map $\beta_{JJ'} : B_J \to B_{J'}$ defined by $b_1 \otimes \cdots \otimes b_n \mapsto b_1 \otimes \cdots \otimes b_n \otimes 1 \otimes \cdots \otimes 1$. Clearly $\beta_{JJ'}$ are A-algebra homomorphism.

Let $\beta_J : B_J \to B$ denote the canonical homomorphism associated to the direct sum. For any $a \in A, b \in B$, if $b = \beta_J(b_J)$ we define $ab = \beta_J(ab_J)$. Since $\beta_{JJ'}$ are A-algebra homomorphisms the result is independent of the choice of J. Then for any finite $J \subseteq \Lambda$, β_J is a ring homomorphism, and by definition of scalar multiplition over B it is also a A-algebra homomorphism. □